D 100188 **(Pages : 3)** Name.................................

Reg. No..............................

SIXTH SEMESTER UG (CUCBCSS-UG) DEGREE EXAMINATION, MARCH 2024

Mathematics

MAT 6B 12—NUMBER THEORY AND LINEAR ALGEBRA

(2018 Admissions only)

Time : Three Hours Maximum Marks : 120

Section A

Answer **all** *questions.*
Each question carries 1 *mark.*

1. Find gcd (143, 227).
2. Find some integer solutions of $3x + 6y = 18$.
3. Define e-prime.
4. Show that 509 is a prime.
5. Let $n > 1$ be fixed and a, b and c be arbitrary integers. Then prove that $a \equiv b (\bmod n)$ and $b \equiv c (\bmod n)$ implies $a \equiv c (\bmod n)$.
6. $\phi(181) =$ ———.
7. State Wilson's Theorem.
8. Define vector space.
9. Show that every singleton subset $\{x\}$ of a vector space V with $x \neq 0_v$ is linearly independent.
10. Decide whether $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x, y, z) = (y, z, 0)$ is linear.
11. Define range space of a linear mapping.
12. If $f : \mathrm{V} \to \mathrm{W}$ is linear from f is injective if and only if Ker $f =$ ———.

(12 × 1 = 12 marks)

Section B

Answer any **ten** *questions.*
Each question carries 4 *marks.*

13. Using Euclidean Algorithm calculate gcd(10535, 4039).
14. Find the complete solution of the linear Diophantine equation :

    $172x + 20y = 1000.$

    Also find solutions in positive integers, if they exist.

Turn over

15. Prove that the linear Diophantine equation $ax + by = c$ has a solution if and only if $d \mid c$, where $d = \gcd(a, b)$.
16. If p is a prime and $p \mid ab$, then prove that $p \mid a$ or $p \mid b$.
17. Using Sieve of Eratosthenes find all primes not exceeding 100.
18. Show that 41 divides $2^{20} - 1$.
19. Find the remainder obtained when 5^{38} is divided by 11.
20. Find the lcm of 12560 and 5030.
21. Investigate how many times the prime 3 appears in 9!.
22. Define vector space.
23. Prove that any line L that passes through the origin is a subspace of $\mathbb{R}^2$.
24. If the vector space $\mathbb{R}^4$ let :

    A = Span {(1, 2, 0, 1), (– 1, 1, 1, 1) ;

    B = Span {(0, 0, 1, 1), (2, 2, 2, 2)}.

    Determine $A \cap B$.
25. Determine whether the subset $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \right\}$ of $\mathrm{Mat}_{3x1}\mathbb{R}$ is linearly dependent.
26. If $f : \mathbb{R} \to \mathbb{R}$ is linear and $\upsilon : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $\upsilon(x, y) = (x, y - f(x))$ prove that υ is an isomorphism.

(10 × 4 = 40 marks)

Section C

Answer any **six** *questions.*
Each question carries 7 marks.

27. If $k > 0$, then prove that $\gcd(ka, kb) = k \gcd(a, b)$.
28. State and Prove Euclid's Lemma.
29. Prove that $5^{2n+2} - 24n - 25$ is divisible by 576.
30. For each positive integer $n \geq 1$, prove that $n = \sum_{d|n} \phi(d)$ the sum being extended over all positive divisors of n.
31. Find the last two digits in the decimal representation of 3^{256}.
32. Prove that the intersection of any two subspaces of a vector space V is a subspace of V.

33. Let S_1 and S_2 be non-empty subsets of a vector space such that $S_1 \subseteq S_2$. Prove that

    (a) If S_2 is linearly independent then so is S_1 ;

    (b) If S_1 is linearly dependent then so is S_2.

34. Examine that the linear mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$f(x, y, z) = (x + z, x + y + 2z, 2x + y + 3z)$$

is neither surjective nor injective.

35. Let V and W be vector spaces of finite dimension over a field F. If $f : V \to W$ is linear then prove that $\dim V = \dim \operatorname{Im} f + \dim \operatorname{Ker} f$.

(6 × 7 = 42 marks)

Section D

Answer any **two** *questions.*
Each question carries 13 *marks.*

36. If $\gcd(a, b) = d$, then prove that $\gcd\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

37. State and prove Chinese Remainder Theorem.

38. Prove that a non-empty subset S of a vector space V is a basis of V if and only if every element of V can be expressed in a unique way as a linear combination of elements of S.

(2 × 13 = 26 marks)