D 50198 **(Pages : 5)** **Name..**

Reg. No...................................

FIFTH SEMESTER (CUCBCSS—UG) DEGREE EXAMINATION NOVEMBER 2023

Mathematics

MAT 5B 08—DIFFERENTIAL EQUATIONS

(2018 Admissions)

Time : Three Hours Maximum : 80 Marks

Section A

Answer all the **twelve** *questions.*
Each question carries 1 *mark.*

1. Solve the initial value problem $dy/dt = -y + 5, y(0) = y_0$.
2. Verify that $y = e^t$ is a solution of the differential equation $y'' - y = 0$.
3. Find the order of the differential equation $y'' + 7y' + 5y = 0$.
4. Show that $(2x+3) + (2y-2)y' = 0$ is an exact equation.
5. Find the Wronskian of the functions $y_1(t) = \cos t, y_2(t) = \sin t$.
6. Write the characteristic equation of the differential equation $y'' + y' + y = 0$.
7. Transform the equation $u'' + 0.125u' + u = 0$ in to a system of first order equations.
8. Find $\mathcal{L}\left(t^2 + 1\right)$.
9. Define the Heaviside function.
10. Find $\mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right)$.
11. Find the fundamental period of the function $f(x) = \sin 3x$.
12. Show that $f(x) = x^2 \cos 4x$ is an even function.

(12 × 1 = 12 marks)

Turn over

Section B

Answer any **ten** *out of fourteen questions.*
Each question carries 4 marks.

13. Find the temperature $u(x, t)$ at any time in a metal rod 50 cm long, insulated on the sides, which initially has a uniform temperature of 20° throughout and whose ends are maintained at 0° for all time $t \geq 0$.

14. Show that the wave equation $a^2 u_{xx} = u_{tt}$ can be reduced to the form $u_{\zeta\eta} = 0$ by the change of variables $\zeta = x - at, \eta = x + at$.

15. Determine the co-efficients in the Fourier Series of $f(x)$ where $f(x) = \begin{cases} -x, & -2 \leq x < 0 \\ x, & 0 \leq x < 2 \end{cases}$ and $f(x+4) = f(x)$.

16. Using Convolution Integral, find the inverse transform of $H(s) = \dfrac{s}{s^2\left(s^2 + a^2\right)}$.

17. Define the unit impulse function δ and show that $\mathcal{L}(\delta(t)) = 1$.

18. Consider the function $f(t) = \begin{cases} 2, & 0 \leq t < 4, \\ 5, & 4 \leq t < 7, \\ -1, & 7 \leq t < 9, \\ 1, & t \geq 9. \end{cases}$

    Express $f(t)$ in terms of $u_c(t)$.

19. Prove the following : If $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots\ldots, \mathbf{x}^{(n)}$, are solutions of $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on the interval $\alpha < t < \beta$, then in this interval $W\left[\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots\ldots, \mathbf{x}^{(n)}\right]$ either is identically zero or else never vanishes.

20. Given that $y_1(t) = t^{-1}$ is a solution of $2t^2 y'' + 3ty' - y = 0, t > 0$, find a fundamental set of solutions.

21. Find a general solution of the initial value problem $y'' + y' + 9.25y = 0, y(0) = 2, y'(0) = 8$.

22. State The Existence and Uniqueness Theorem.

23. Solve the differential equation $\left(y\cos x + 2xe^{y}\right) + \left(\sin x + x^{2}e^{y} - 1\right)y' = 0$.

24. Solve the initial value problem $y' = y^{2}, y(0) = 1$ and determine the interval in which the solution exists.

25. Show that the equation $y'' = \dfrac{x^{2}}{1 - y^{2}}$ is separable, and then find an equation for its integral curves.

26. Solve $\dfrac{dy}{dx} = \sin(x + y)$.

(10 × 4 = 40 marks)

Section C

Answer any **six** *out of nine questions.*
Each question carries 7 marks each.

27. (a) Let $y = y_{1}(t)$ is a solution of $y' + p(t)y = 0$ and let $y = y_{2}(t)$ be a solution of $y' + p(t)y = g(t)$. Show that $y(t) = y_{1}(t) + y_{2}(t)$ is also a solution of $y' + p(t)y = g(t)$.

    (b) Find the value of b for which he given equation $\left(xy^{2} + bx^{2}y\right)dx + (x + y)x^{2}dy = 0$ is exact.

28. Find an integrating factor for the equation $\left(3xy + y^{2}\right) + \left(x^{2} + xy\right)y' = 0$ and hence solve it.

29. Find a particular solution of $y'' - 3y' - 4y = 3e^{2t} + 2\sin t$.

30. Suppose that a mass of weighing 10 lb stretched in a spring 2 in. If the mass is displaced an additional 2 in. and is then set in motion with an initial upward velocity of 1 ft/s, determine the position of the mass at an later time. Also determine the period , amplitude and phase of the potion.

31. If the function f defined by $f(t) = \begin{cases} \sin t, 0 \le t < \pi/4, \\ \sin t + \cos(t - \pi/4), t \ge \pi/4 \end{cases}$.

    Find $\mathcal{L}(t)$.

Turn over

32. (a) Show that if c is a positive constant, then $\mathcal{L}(ct) = \frac{1}{c} F(s/a), s > ca.$

(b) Show that if a and b are constants with $a > 0$, then $\mathcal{L}^{-1} F[(as+b)] = \frac{1}{a} e^{-bt/a} f\left(\frac{t}{a}\right).$

33. State and prove The Convolution Integral theorem for Laplace Transform.

34. Find a Fourier sine series for $f(x) = \begin{cases} x, 0 < x < \pi/2 \\ \pi - x, \pi/2 < x < \pi \end{cases}$.

35. Show that $\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx = \begin{cases} 0, m \neq n \\ L, m = n \end{cases}$.

(6 × 7 = 42 marks)

Section D

Answer any **two** *questions.*
Each question carries 13 *marks each.*

36. (a) Consider a vibrating string of length L = 30 that satisfy the wave equation $4u_{xx} = u_{tt}, 0 < x < 30, t > 0.$ Assume that the ends of the string are fixed and that the string is set in motion with no initial velocity from the initial position

$$u(x,0) = f(x) = \begin{cases} x/10, 0 \le x \le 10, \\ (30-x)/20, 10 < x \le 30. \end{cases}$$

Find the displacement $u(x,t)$ of the string and describe its motion through one period.

(b) Find the Fourier series for the function $f(x) = \begin{cases} x, 0 < x < \pi \\ 2\pi - x, \pi < x < 2\pi \end{cases}$ and hence show that

$$\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

37. (a) Using Laplace transform, find the solution of the initial value problem $y'' + y = \sin 2t,\ y(0) = 2,\ y'(0) = 1.$

(b) Find the inverse Laplace Transform of $\dfrac{1}{s\left(2s^2 + s + 2\right)}$.

38. (a) State and Prove Abel's Theorem.

(b) Using the Method of Variation of Parameters, solve $y'' + 4y = 3csct.$

(2 × 13 = 26 marks)