D 50666 **(Pages : 3)** **Name..**

Reg. No....................................

FIFTH SEMESTER (CBCSS—UG) DEGREE EXAMINATION NOVEMBER 2023

Mathematics

MTS 5B 06—BASIC ANALYSIS

(2020 Admission onwards)

Time : Two Hours and a Half — Maximum : 80 Marks

Section A

Answer any number of questions.
Each question carries 2 marks.
Maximum 25 marks.

1. State Cantor's theorem.
2. Prove that there does not exist a rational number r such that $r^2 = 3$.
3. If $a, b \in \mathbb{R}$, prove that $\big||a| - |b|\big| \leq |a - b|$.
4. Prove that an upper bound u of a nonempty set S in $\mathbb{R}$ is the supremum of S if and only if for every $\in > 0$ there exist an $s_{\in} \in S$ such that $u - \in < s_{\in}$.
5. State and prove Archimedean property.
6. Prove that a sequence in $\mathbb{R}$ can have at most one limit point.
7. Prove that $\lim \left(\frac{\sin n}{n}\right) = 0$.
8. Let $e_n = \left(1 + \frac{1}{n}\right)^n$ for $n \in \mathbb{N}$. Prove that $2 < e_n < 3$ for all $n \in \mathbb{N}$.
9. Give an example of an unbounded sequence that has a convergent subsequence.
10. If (x_n) and (y_n) are Cauchy sequences, prove that $(x_n + y_n)$ is a Cauchy sequence.

Turn over

11. Prove that the sequence $\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}\right)$ diverges.

12. Define Cantor set.

13. Describe the set of points z in the complex plane that satisfy the equation $|z-2| = \text{Re}(z)$.

14. Find the image of the line segment from 1 to i under the complex mapping $\omega = \overline{iz}$.

15. Find the image of the rectangle with vertices $-1+i,\ 1+i,\ 1+2i$ and $-1+2i$ under the linear mapping $f(z) = 4iz + 2 + 3i$.

Section B

Answer any number of questions.
Each question carries 5 *marks.*
Maximum 35 *marks.*

16. Prove that the following statements are equivalent :

    a) S is a countable set ; and

    b) There exists a surjection of $\mathbb{N}$ onto S.

    c) There exists an injection of S onto $\mathbb{N}$.

17. Determine the set $B = \{x \in \mathbb{R} : |x-1| < |x|\}$.

18. Prove that $\mathbb{R}$ of real numbers is not countable.

19. Let $X = (x_n)$ and $Y = (y_n)$ be sequences of real numbers that converge to x and y respectively. Prove that $X \cdot Y$ converges to xy.

20. State and prove Cauchy convergence criterion for sequences.

21. Prove that

    a) the union of an arbitrary collection of open subsets in $\mathbb{R}$ is open.

    b) the intersection of any finite collection of open sets in $\mathbb{R}$ is open.

22. Determine whether the points $z_1 = -2 - 8i, z_2 = 3i, z_3 = -6 - 5i$ are the vertices of a right triangle.

23. Let $S = \{z \in \mathbb{C} : 1 \le |z - 1 - i| < 2\}$. Determine whether the set S is :

    a) Open ;

    b) Closed ;

    c) Domain ;

    d) Bounded ; and

    e) Connected.

Section C

Answer any **two** *questions.*
Each question carries 10 *marks.*

24. (a) Let a and b be positive real numbers, prove that $\sqrt{ab} \le \frac{1}{2}(a + b)$ and the equality holds if and only if $a = b$.

    (b) State and prove Bernoulli's inequality.

25. Prove that there exits a positive real number x such that $x^2 = 2$.

26. (a) Prove that every contractive sequence is a convergent sequence.

    (b) The polygonal equation $x^3 - 7x + 2 = 0$ has a root r with $0 < r < 1$. Use an appropriate contractive sequence to calculate r within 10^{-5}.

27. (a) Solve the simultaneous equations $|z| = 2$ and $|z - 2| = 2$.

    (b) Find the image of the triangle with vertices $0, 1 + i$ and $1 - i$ under the mapping $\omega = z^2$.

(2 × 10 = 20 marks)