D 100186 **(Pages : 4)** **Name..................................**

Reg. No...............................

SIXTH SEMESTER UG (CUCBCSS-UG) DEGREE EXAMINATION, MARCH 2024

Mathematics

MAT 6B 01—COMPLEX ANALYSIS

(2018 Admissions only)

Time : Three Hours Maximum Marks : 120

Section A

Answer **all** *questions.*
Each question carries 1 mark.

1. For the complex numbers $z_0 = a + ib$ and $z_1 = c + id$, $\lim\limits_{z \to z_0} z_1 =$ ________.

2. State True or False : The function

$$f(z) = \begin{cases} \dfrac{\operatorname{Re} z}{|z|}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is continuous at $z = 0$.

3. Find the derivative of $f(z) = z^2$.

4. Compute the principal value of $\log_e z$ when $z = 1 + i$.

5. Find all the roots of the equation $\tan z = 1$.

6. Evaluate $\int_i^{1+4i} z^2 dz$.

7. If C is the simple closed contour given by the circle $|z| = 2$, then $\int_C dz =$ ________.

8. Integrate $\dfrac{z^2+1}{z^2-1}$ in the contour clock wise sense around a circle of radius 1 with centre at the point $z = \dfrac{1}{2}$.

9. State Maximum modulus principle.

10. If the sequence $\sqrt[n]{|a_n|}, n = 1, 2, \ldots$ converges with the limit L > 0, then the radius of convergence of the power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \ldots$$

is ________.

Turn over

11. Define removable singular point.
12. State Cauchy's Residue Theorem.

(12 × 1 = 12 marks)

Section B

Answer any **ten** *questions.*
Each question carries 4 marks.

13. Prove that if the limit of the function $f(z)$ exists at a point z_0, then it is unique.
14. Verify Cauchy-Riemann equations for the function f given by $f(z)=\frac{x-iy}{x^2+y^2}$.
15. Examine the differentiability at the origin of the function f given by $f(z)=|z|^2$.
16. Examine the analyticity of $f(z)=\cosh x\cos y+i\sinh x\sin y$.
17. Find the real part of e^{-3z}.
18. Prove that $\sinh(z_1+z_2)=\sinh z_1\cosh z_2+\cosh z_1\sinh z_1$.
19. Evaluate $\int_0^1 (1+it)^2\,dt$.
20. If $f'(z)=0$ everywhere in a domain D, then prove that $f(z)$ must be constant throughout D.
21. If M be any non-negative cosntant such that $|f(z)|\le M$ everywhere on a contour C and L is the length of C, then prove that $\left|\int_C f(z)dz\right|\le ML$.
22. Evalaute $\int_C \frac{dz}{z-a}$, where C is the circle $|z-a|=r$ oriented in the positive direction.
23. Examine the convergence of the series $\sum_{n=1}^{\infty}\frac{z^n}{n^2}$ on its circle of convergence.
24. Find the expansion for $f(z)=z^2e^z$.
25. What kind of singularity the function $\frac{\cot\pi z}{(z-a)^2}$ has at the point $z = a$.
26. Find the residue of the function $f(z)=\frac{z}{z^4+4}$ at the isolated singular point $1 + i$.

(10 × 4 = 40 marks)

Section C

Answer any **six** *questions.*
Each question carries 7 *marks.*

27. If $w = f(z) = \bar{z}$, show that $\frac{dw}{dz}$ does not exist at the origin.

28. Evaluate $\oint_C \frac{dz}{z^2 + 9} dz$, where C is the unit circle.

30. Using principle of deformation of paths, evaluate $\int_C \frac{1}{z} dz$ where C is any positively oriented closed contour surrounding the origin.

31. Find an analytic function whose real part $e^x (x \cos y - y \sin y)$ and which takes the value e at $z = 1$.

32. If R_1 and R_2 are the radius of convergences of the power series $\sum_{n=0}^{\infty} a_n z^n$ and $\sum_{n=0}^{\infty} b_n z^n$ respectively, show that the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n b_n z^n$ is $R_1 R_2$.

33. Find series representation of $f(z) = \frac{-1}{(z-1)(z-2)}$.

34. If C is a simple closed contour containing the origin, show that $\frac{1}{2\pi i} \int_C \frac{e^{az}}{z^{n+1}} dz = \frac{a^n}{n!}$.

35. Show that if $s > 0, a > 0,$ then prove that $\int_{-\infty}^{\infty} \frac{e^{1sx}}{x^2 + a^2} dx = \frac{\pi}{a} e^{-as}$.

(6 × 7 = 42 marks)

Section D

Answer any **two** *questions.*
Each question carries 13 *marks.*

36. (a) Find the Laurent series of $f(z) = \frac{1}{1 - z^2}$ with centre at $z = 1$.

(b) If R be the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$ what is the radius of convergence of $\sum_{n=0}^{\infty} a_n^2 z^n$.

Turn over

37. (a) Evaluate $\oint_C \sec z\, dz$, where C is the unit circle.

(b) If $w = f(z) = \bar{z}$, show that $\frac{dw}{dz}$ doesn't exist at any point.

38. (a) Prove by contour integration that $\int_0^{2\pi} e^{\cos\theta} \cos(n\theta - \sin\theta)\, d\theta = \frac{2\pi}{n!}$ (n positive integer).

(b) Show that $\int_{-\infty}^{\infty} \frac{\cos 3x}{\left(x^2+1\right)^2}\, dx = \frac{2\pi}{e^3}$.

(2 × 13 = 26 marks)