D 10231 (Pages : 3) Name....................................

Reg. No..................................

FIFTH SEMESTER B.A./B.Sc. DEGREE EXAMINATION, NOVEMBER 2021

(CUCBCSS–UG)

Mathematics

MAT 5B 06—ABSTRACT ALGEBRA

Time : Three Hours Maximum : 120 Marks

Part A

Answer **all** *questions.*
Each question carries 1 *mark*.

1. The smallest non abelian group has ———— number of elements.
2. The order of the identity element in any group G is ————.
3. State True or False. "Every abelian group is cyclic".
4. State True or False. "Every group of order 31 is cyclic".
5. Give an example of non-cyclic group with four elements.
6. The total number of subgroups of Z_{12} is ————.
7. What are the orbits of the identity permutation σ of a set A ?
8. How many zero divisors are there for the field Z_7 ?
9. How many units are there for the field Z_7 ?
10. Give an example of integral domain which not a field.
11. State True or False. Z is a sub field of Q.
12. Write the number of generators of the group Z_5 under addition modulo 5.

(12 × 1 = 12 marks)

Part B

Answer any **ten** *questions.*
Each question carries 4 *marks*.

13. Show that left and right cancellation holds in a group G.
14. Let G be a group and suppose that $a * b * c = e \forall a, b, c \in$ G. Show that $b * c * a = e$.
15. Prove that a group G has exactly one idempotent element.
16. Can the identity element be a generator of a cyclic group ?
17. Prove that every cyclic group is abelian.

Turn over

18. Consider the group $\mathbb{Z}_{12}$, under the operation addition modulo 12. Find the order of the cyclic subgroup generated by $3 \in \mathbb{Z}_{12}$.
19. Show that the permutation (1, 4, 5, 6) (2, 3, 1, 5) is an even permutation.
20. What is the order of the cycle (1, 4, 5, 7) in S_8 ?
21. Find the partition of the group $\mathbb{Z}_6$, under the operation addition modulo 6, into cosets of the subgroup H = {0, 3}.
22. Consider the rings $\langle \mathbb{Z}, +, \cdot \rangle$ and $\langle 2\mathbb{Z}, +, \cdot \rangle$. Verify whether the map $\phi : \mathbb{Z} \to 2\mathbb{Z}$ defined by $\phi(x) = 2x \forall x \in \mathbb{Z}$ is a ring homomorphism or not.
23. Find the number of generators of the cyclic group of order 8.
24. Solve the equation $x^2 - 5x + 6 = 0$ in $\mathbb{Z}_{12}$.
25. Consider the following two binary structures :

    (a) $\mathbb{Z}$, the set of integers with ordinary addition ; and

    (b) $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ the set of even integers with ordinary addition.

    Show that the above two binary structures are isomorphic.
26. Let n be a positive integer. Give an example of a group containing n elements.

(10 × 4 = 40 marks)

Part C

Answer any **six** *questions.*
Each question carries 7 *marks.*

27. Let $*$ be defined by Q^+ by $a * b = \dfrac{ab}{2}$. Show that $\left(Q^+, *\right)$ is an abelian group.
28. Let G be a group. For all $a, b \in G$, prove that $(a * b)^{-1} = b^{-1} * a^{-1}$.
29. Prove that a necessary and sufficient condition that a non-empty subset H of a group G is a subgroup of G is that $a \in H, b \in H \Rightarrow ab^{-1} \in H$.
30. Show that the subgroups of $\mathbb{Z}$ under addition are precisely the groups $n\mathbb{Z}$ under addition for $n \in \mathbb{Z}$.
31. Show that any permutation of a finite set of at least two elements is a product of transpositions.
32. Show that a homomorphism ϕ of a group G is a one-to-one function if and only if Ker $\phi = \{e\}$.
33. Show that cancellation law holds in a ring R if and only if R has no zero divisors.

34. M_2 denotes the ring of all 2×2 matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are rational numbers. Is M_2 a field ? Justify your answer.

35. Prove that any integral domain D can be enlarged to (or embedded in) a field F such that every element of F can be expressed as a quotient of two elements of D.

(6 × 7 = 42 marks)

Part D

Answer any **two** *questions.*
Each question carries 13 *marks.*

36. Show that subgroup of a cyclic group is cyclic.

37. (a) Define the term orbit, cycle and transposition with respect to a permutation.

 (b) Write the permutation $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 5 & 2 & 4 & 3 & 1 \end{pmatrix}$ as product of disjoint cycles.

 (c) Define even and odd permutation. Write (1, 4, 5, 6) (2, 1, 5) as a product of transpositions.

38. Show that in the ring $\mathbb{Z}_n$, the divisors of 0 are precisely those elements that are not relatively prime to n also show that $\mathbb{Z}_p$ is a field.

(2 × 13 = 26 marks)