**D 30178** **(Pages : 3)** **Name.......................................**

**Reg. No..................................**

# FIFTH SEMESTER (CUCBCSS—UG) DEGREE EXAMINATION NOVEMBER 2022

Mathematics

MAT 5B 06—ABSTRACT ALGEBRA

(2017—2018 Admissions)

Time : Three Hours Maximum : 120 Marks

## Section A

*Answer* **all twelve** *questions.*
*Each question carries* 1 *mark.*

1. Whether usual addition a binary operation on the set of all prime numbers ? Justify your answer.
2. State True *or* False. "Every cyclic group is abelian".
3. Give an example of non-abelian group.
4. Number of elements in the group $S_3$ is ————.
5. Define Kernel of a homomorphism.
6. The total number of subgroups of $Z_{12}$ is ————.
7. State theorem of Lagrange.
8. What are the zero divisors of $Z_6$ ?
9. How many unit elements are there in the ring Z ?
10. What is the characteristic of $Z_5$.
11. State true *or* false : "Every field is an integral domain".
12. Give an example of ring which is not a field.

(12 × 1 = 12 marks)

## Section B

*Answer any* **ten** *out of fourteen questions.*
*Each question carries* 4 *marks.*

13. Show that identity element is unique in a group G with binary operation *.
14. Check whether the binary operation * defined on Z by $a*b = ab + 1$ is commutative and associative.

**Turn over**

15. Let S be a set consisting of 10 people, no two of whom are of the same height. Define * by $a*b = c$, where $c$ is the shortest person in S who is taller than both $a$ and $b$. Is * $a$ binary operation ?

16. If G be a group with identity e such that $x^2 = e$ for all element $x$ in G. Then show that G is abelian.

17. Write at least five elements of the cyclic group 25Z under addition.

18. Can the identity element be a generator of a cyclic group ?

19. Show that the permutation (1, 4, 5, 6) (2, 1, 5) is an odd permutation.

20. Prove that every permutation $\sigma$ of a finite set is a product of disjoint cycles.

21. State and prove Lagrange's theorem.

22. Prove that if in a ring R, $(a + b)^2 = a^2 + 2ab + b^2$ for all $a, b \in \mathbb{R}$, then the ring is commutative

23. Find the number of generators of the cyclic group of order 10.

24. Prove that every field F is an integral domain.

25. Consider the following two binary structures :

    (a) $\mathbb{Z}$, the set of integers with ordinary addition and

    (b) $2\mathbb{Z} = \{2n \mid n \in \mathbb{Z}\}$ the set of even integers with ordinary addition.

    Show that the above two binary structures are isomorphic.

26. Let $n$ be a positive integer. Give an example of a group containing $n$ elements.

(10 × 4 = 40 marks)

## Section C

*Answer any* **six** *out of nine questions.*
*Each question carries 7 marks.*

27. Let G be the set of all real numbers a $\neq -1$ with a binary operation * on G defined by

    $a * b = a + b + ab \; \forall \; a, b \in$ G.

    Prove that G is an abelian group under the operation*.

28. Show that intersection of sub groups $H_1$ of a group G for $i \in I$ is again a sub group of G. What about union of two sub groups ?

29. Give an example of an infinite group which is not cyclic.

30. What is the order of $\sigma = (1, 4)\,(3, 5, 7, 8)$ in $S_8$ ?

31. Compute the indicated product of cycles that are permutations of {1, 2, 3, 4, 5, 6, 7, 8}.

    a) (1, 4, 5) (7, 8) (2, 5, 7).

    b) (1, 3, 2, 7) (4, 8, 6).

32. Show that every subgroup of an abelian group is normal subgroup.

33. Show that $\phi : G \rightarrow G'$ where G and $G'$ are two groups, defined by $\phi(g) = e'$ for $g \in G$ is a homomorphism.

34. If R is a ring with unity, then show that this unity 1 is the only multiplicative identity.

35. Let G be the group of non-zero complex numbers under multiplication and let $n$ be any positive integer. Show that the mapping $f : G \rightarrow G$ defined by $f(z) = z^n$ is a homomorphism. What is the kernel of this homomorphism ?

(6 × 7 = 42 marks)

## Section D

*Answer any* **two** *out of three questions.*
*Each question carries* 13 *marks.*

36. Let $\phi$ be a homomorphism of a group G into a group G'. Then prove the following :

    a) If $e$ is the identity in G, then $\phi(e)$ is the identity e' in G'.

    b) If $a \in G$, then $\phi(a^{-1}) = \phi(a)^{-1}$.

    c) If H is a subgroup of G, then $\phi$ [H] is a subgroup of G'.

    d) If K' is a subgroup of G', then $\phi^{-1}$ [K'] is a subgroup of G.

37. Show that for $n \geq 2$, the number of even permutations in $S_n$ is the same as the number of odd permutations.

38. (a) State and prove Lagrange's theorem.

    (b) Prove that the order of an element of a finite group divides the order of the group.

    (c) Find the order of a element 2 in the group $(Z_5, +)$.

(2 × 13 = 26 marks)